Speeds of coming down from infinity for continuous-state nonlinear branching processes

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The 14th International Workshop on Markov Processes and Related Topics
July 19, 2018, Chengdu

Outline of the Talk

1. Introduction
   - Continuous-state branching process
   - Nonlinear continuous-state branching process
   - Elements of SNLP
2. Main Results
   - Moments of the hitting time for nonlinear CSBP
   - Laplace transform of the hitting time
   - Speeds of coming down from infinity

Feller branching process and Branching property

- A Feller branching process is the unique nonnegative solution to SDE
$$X_t = X_0 + \int_0^t \sqrt{\gamma X_s} \mathrm{d}B_s^X,$$
where B^X is a Brownian motion and $\gamma > 0$ is the branching rate.
- Let Y be another independent Feller branching processes.
$$Y_t = Y_0 + \int_0^t \sqrt{\gamma Y_s} \mathrm{d}B_s^Y$$
Since $X + Y$ is a continuous martingale with $\langle X + Y \rangle_t = \langle X \rangle_t + \langle Y \rangle_t = \int_0^t \gamma(X_s + Y_s)\mathrm{d}s$, then $X + Y$ solves SDE $Z_t = X_0 + Y_0 + \int_0^t \sqrt{\gamma Z_s} \mathrm{d}B_s^Z$.
- Such an additive property is called branching property.

Continuous-state branching process (CB processes)

- Feller diffusion is an example of continuous-state branching processes. In general, a continuous-state branching process is a nonnegative Markov process X (with no negative jumps) satisfying the branching property.
- Its Laplace transform is determined by

$$\mathbb{E}_x e^{-\theta X_t} = e^{-x u_t(\theta)}$$

where function $u_t(\theta)$ satisfies the differential equation

$$\frac{\partial u_t(\theta)}{\partial t} + \psi(u_t(\theta)) = 0$$

with $u_0(\theta) = \theta$ and

$$\psi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x)\Pi(\mathrm{d}x)$$

for $\sigma > 0$, $b \in \mathbb{R}$ and for σ-finite measure Π on $(0, \infty)$ satisfying $\int_0^\infty (z \wedge z^2)\Pi(\mathrm{d}z) < \infty$.

Lamperti transform

- CB process is associated with a spectrally positive Lévy process via the Lamperti time change.
- Write Z for a spectrally positive Lévy process and $\tau_0^- := \inf\{t : Z_t = 0\}$. for its first time of reaching 0. Write

$$\eta(t) := \int_0^{t\wedge\tau_0^-} \frac{1}{Z_s} ds \quad \text{and} \quad \eta^{-1}(t) := \inf\{s \geq 0 : \eta(s) > t\}.$$

Then process $X_t := Z_{\eta^{-1}(t)\wedge\tau_0^-}$ is the CB process.

A class of nonlinear continuous-state branching processes

- We can generalize the above mentioned Lamperti transform. Let R be a positive function on $[0, \infty)$. Define

$$\eta(t) := \int_0^{t\wedge\tau_0^-} \frac{1}{R(Z_s)} ds \quad \text{and} \quad \eta^{-1}(t) := \inf\{s \geq 0 : \eta(s) > t\}.$$

  Then $X_t := Z_{\eta^{-1}(t)\wedge\tau_0^-}$ is a nonlinear continuous-state branching process with branching rate function $R(x)$.
- Process X has a generator L on $C^2[0, \infty)$ such that

$$Lf(x) := R(x)L^* f(x)$$
$$:= R(x)\left(-bf'(x) + \frac{\sigma^2}{2} f''(x) + \int_0^\infty (f(x+u) - f(x) - uf'(x))\right) \Pi$$

  where L^* is the generator for a spectrally positive Lévy process.

- Intuitively, it is a branching process whose branching rate depends on the current population size.
- Observe that for $b > 0$, $\tau_b^- := \inf\{t : Z_t < b\}$ and $T_b^- := \inf\{t : X_t < b\}$,

$$T_b^- = \int_0^{\tau_b^-} 1/R(Z_s)ds.$$

We need to understand the weighted occupation time for Z.
- Such a nonlinear continuous-state branching process had been studied in P. Li (2016).

- A spectrally positive Lévy process (SPLP) is a Lévy process with no negative jumps.
- For spectrally positive Lévy process Z_t with $Z_0 = 0$,

$$\mathbb{E}\mathrm{e}^{-\theta Z_t} = \mathrm{e}^{t\psi(\theta)},$$

for $\theta, t \geq 0$, where the Laplace exponent for $-Z$ is

$$\psi(\theta) = b\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{0} \left(\mathrm{e}^{\theta z} - 1 - \theta z\right) \Pi(\mathrm{d}z),$$

for $\gamma \in \mathbb{R}$ and $\sigma \geq 0$. Also, the Lévy measure Π is a σ-finite measure on $(-\infty, 0)$ such that

$$\int_{-\infty}^{0} (z \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

- $\mathbb{E}Z_1 = -\psi'(0+)$.

Scale function

- The Laplace exponent ψ is strictly convex and $\lim_{\theta\to\infty}\psi(\theta)=\infty$. Thus, there exists an inverse function $\Phi\colon[0,\infty)\to[0,\infty)$ such that

$$\psi(\Phi(\theta))=\theta,\quad \theta\geq 0.$$

- We often need scale functions to study the fluctuations of SNLP.
- The scale function W of the process $-Z$ is defined as the function with Laplace transform on $[0,\infty)$ given by

$$\int_0^\infty \mathrm{e}^{-\lambda z}W(z)\mathrm{d}z=\frac{1}{\psi(\lambda)}\quad\text{for } \lambda>\Phi(0).$$

Coming down from infinity

- We say the process X comes down from infinity if
$$\lim_{b\to\infty}\lim_{x\to\infty}\mathbb{P}_x(T_b^- < \infty) = 1. \tag{1}$$
- Let $\mathbb{P}_\infty := \lim_{x\to\infty}\mathbb{P}_x$ and $\mathbb{E}_\infty$ be the corresponding expectation.
- We can show that X comes down from infinity if and only if $\mathbb{E}_\infty T_b^- < \infty$ for all large b.
- If X comes down from infinity, then ∞ can be treated as an entrance boundary. $\mathbb{P}_\infty$ is the entrance law.
- The speed of coming down from infinity has been studied for Λ-coalescent in Berestycki et al (2010) and for birth and death processes in Bansaye et al (2015).

Why coming down from infinity?

- If process Z has a negative drift, it can go down by a large amount over a long period.
- Starting at x with a large value of $R(x)$, due to the time change, a short time behavior of X near time 0 corresponds to a long term behavior of Z, and we see a significant drop for X over the short time period;
- If X is subcritical and $R(x)$ increases fast enough as $x \to \infty$, the drop becomes so drastic and coming down from infinity occurs;
- If X is critical, coming down from infinity can still occur if $R(x)$ increases even faster as $x \to \infty$.

We are interested in two kinds of rate function $R(x)$.

- (slow regime) $R(x) = x^\theta, \ x, \theta > 0$.
- (fast regime) $R(x) = e^{\theta x}, \ x, \theta > 0$.

Outline of the strategy

Assume that X comes down from infinity.

- Under $\mathbb{P}_\infty$ we first estimate T_b^- as $b \to \infty$.
- Let $\underline{X}_t := \inf_{0\le s\le t} X_s$ be the running minimum process of X. Since $\underline{X}_{T_b^-} = b$, we can show that $\lim_{t\to 0+} \frac{\underline{X}_t}{g^{-1}(t)} = 1$ in probability, where g^{-1} is the inverse function of $g(x) := \mathbb{E}_\infty T_x^-$.
- If X is subcritical, we can show that $\lim_{t\to 0+} \frac{\underline{X}_t}{X_t} = 1$, and we have $\frac{X_t}{g^{-1}(t)} = 1$ in probability as $t \to 0+$.

- Recall that $T_b^- = \int_0^{\tau_b^-} 1/R(Z_s)ds$.
- A local time for the process Z is its occupation density.
- Write $l(a,t)$ for the local time of Z at level a up to time t.
- Given $\psi'(0+) \geq 0$, by B. Li and Z. (2017) we have for $x, a > b > 0$,

$$\mathbb{E}_x l(a, \tau_b^-) = W(a-b) - W(a-x)$$

and for $x, a_1, a_2 > b > 0$,

$$\begin{aligned}&\mathbb{E}_x[l(a_1,\tau_b^-)l(a_2,\tau_b^-)]\\&= [W(a_2-b) - W(a_2-x)][W(a_1-b) - W(a_1-a_2)]\\&\quad + W(a_2-b)[W(a_1-b) - W(a_1-x)].\end{aligned}$$

The moments of T_b^-

Suppose that $\psi'(0+) \geq 0$. Then for $x > b > 0$,

$$\begin{aligned}\mathbb{E}_x(T_b^-) &= \mathbb{E}_x \int_0^\infty R^{-1}(X_s) 1_{s<\tau_b^-} ds \\ &= \int_b^\infty R^{-1}(y) \mathbb{E}_x l(y, \tau_b^-) dy \\ &= \int_b^\infty R^{-1}(y)[W(y-b) - W(y-x)] dy.\end{aligned}$$

$$\mathbb{E}_\infty(T_b^-) = \int_b^\infty R^{-1}(y) W(y-b) dy.$$

X comes down from infinity if and only if $\int_b^\infty R^{-1}(y) W(y-b) dy < \infty$ for b large enough.

$$\mathbb{E}_x(T_b^{-2}) = \mathbb{E}_x\left(\int_0^\infty R^{-1}(X_s)1_{s<\tau_b^-}ds\right)^2$$
$$= \int_b^\infty R^{-1}(a_1)da_1 \int_b^\infty R^{-1}(a_2)\mathbb{E}_x l(a_1,\tau_b^-)l(a_2,\tau_b^-)da_2$$
$$= \int_b^\infty R^{-1}(a_1)da_1 \int_b^\infty R^{-1}(a_2)$$
$$\times [W(a_2-b)-W(a_2-x)][W(a_1-b)-W(a_1-a_2)]da_2.$$

$$\mathbb{E}_\infty(T_b^{-2}) = \int_b^\infty R^{-1}(a_1)da_1 \int_b^\infty R^{-1}(a_2)$$
$$\times W(a_2-b)[W(a_1-b)-W(a_1-a_2)]da_2.$$

If $R(x) = x^\theta$, then

$$\mathbb{E}_\infty(T_b^-) = \int_b^\infty \frac{W(y-b)}{y^\theta} dy = \int_0^\infty \frac{W(y)}{(b+y)^\theta} dy$$

$$= \frac{1}{\Gamma(\theta)} \int_0^\infty W(y) dy \int_0^\infty e^{-\lambda(b+y)} \lambda^{\theta-1} d\lambda$$

$$= \frac{1}{\Gamma(\theta)} \int_0^\infty e^{-\lambda b} \lambda^{\theta-1} \int_0^\infty W(y) e^{-\lambda y} dy d\lambda$$

$$= \frac{1}{\Gamma(\theta)} \int_0^\infty \frac{e^{-\lambda b} \lambda^{\theta-1}}{\psi(\lambda)} d\lambda.$$

If $R(x) = e^{\theta x}$, then $\mathbb{E}_\infty(T_b^-) = \frac{1}{e^{\theta b}\psi(\theta)}$.

An asymptotic result in the slow regime

Proposition

Suppose that X is subcritical and there exist $0 < c_1 < c_2$ and $\theta > 1$ such that

$$c_1 x^\theta \le R(x) \le c_2 x^\theta$$

for all x large enough. Then as $b \to \infty$,

$$\frac{T_b^-}{\mathbb{E}_\infty T_b^-} \to 1 \quad \textit{in probability under } \mathbb{P}_\infty.$$

Outline of the proof: Check that $\mathrm{Var}_\infty \frac{T_b^-}{\mathbb{E}_\infty T_b^-} \to 0$.

Under an additional mild condition on ψ we can show the convergence holds almost surely.

Laplace transform of T_b^-

Lemma

Given a locally bounded nonnegative function R on $(0, \infty)$, let $W_n(x)$ satisfy

$$W_0(x) = e^{-\Phi(0)x}, \ W_{n+1}(x) = \int_x^\infty \frac{W(z-x)}{R(z)} W_n(z)dz, \ x \geq 0, n \geq 0.$$

Given $b \geq 0$, if $\sum_{n=0}^\infty W_n(b) < \infty$, then for all $x \geq b$,

$$\mathbb{E}_x[e^{-\lambda T_b^-}; T_b^- < \infty] = \mathbb{E}_x\left[\exp\left(-\lambda \int_0^{\tau_b^-} \frac{1}{R(Z_s)} ds\right); \tau_b^- < \infty\right]$$

$$= \frac{\sum_{n=0}^\infty \lambda^n W_n(x)}{\sum_{n=0}^\infty \lambda^n W_n(b)}.$$

Proposition

Suppose $R(x) = x^{\theta_1} e^{\theta_2 x}$, $\theta_2 > 0, \theta_1 > 0$, $\psi'(0+) \geq 0$ *and*

$$\mathbb{E}_\infty T_b^- = \int_b^\infty \frac{W(y-b)}{R(y)} dy < \infty.$$

Then the sequence $W_n(x)$, $x > b$, *can be expressed as* $W_0(x) = 1$ *and for* $n \geq 1$,

$$W_n(x) = \frac{1}{\Gamma(\theta_1)^n} \int_{(0,\infty)^n} \frac{e^{-\sum_{i=1}^n (\lambda_i + \theta_2)x} (\lambda_1 \dots \lambda_n)^{\theta_1 - 1}}{\psi(\lambda_1 + \theta_2) \dots \psi(\sum_{i=1}^n (\lambda_i + \theta_2))} d\lambda_1 \dots d\lambda_n. \tag{2}$$

Let L be a generator on $C^2([0,\infty))$ such that

$$
\begin{aligned}
Lf(x) &:= x^{\theta_1} e^{\theta_2 x} L^* f(x) \\
&:= x^{\theta_1} e^{\theta_2 x} \left(-bf'(x) + \frac{\sigma^2}{2} f''(x) + \int_0^\infty (f(x+u) - f(x) - uf'(x)) \right.
\end{aligned}
$$

i.e. L is the generator for the corresponding nonlinear continuous-state branching process with $R(x) = x^{\theta_1} e^{\theta_2 x}$. For $f(x) = e^{-\lambda x}$ we have $L^* f(x) = e^{-\lambda x} \psi(\lambda)$. One can check that for the function W_n given in (2), we have $L^* W_n(x) = x^{-\theta_1} e^{-\theta_2 x} W_{n-1}(x)$. It follows that

$$
L\left(\sum_{i=0}^{\infty} \lambda^n W_n\right)(x) = \lambda \sum_{i=0}^{\infty} \lambda^n W_n(x),
$$

i.e. $\sum_{i=0}^{\infty} \lambda^n W_n$ is a λ-invariant function of L and it is a "scale function" for process X.

Corollary

Suppose $R(x) = x^\theta$, $\theta > 0$, $\psi'(0+) \geq 0$ *and*

$$\int_{0+} \frac{\lambda^{\theta-1}}{\psi(\lambda)} d\lambda < \infty.$$

Then the sequence $W_n(x)$, $x > b$ *is given by* $W_0(x) = 1$ *and*

$$W_n(x) = \frac{1}{\Gamma(\theta)^n} \int_{(0,\infty)^n} \frac{\exp(-\sum_{i=1}^n \lambda_i x)(\lambda_1 \ldots \lambda_n)^{\theta-1}}{\psi(\lambda_1) \ldots \psi(\sum_{i=1}^n \lambda_i)} d\lambda_1 \ldots d\lambda_n, \qquad n$$

In particular, for $\psi(\lambda) = \lambda^\alpha$, $1 < \alpha \leq 2$ *and* $\theta > \alpha$ *we have*

$$W_0(x) = 1, \qquad W_n(x) = x^{n\alpha - n\theta} \prod_{i=1}^n \frac{\Gamma(i\theta - i\alpha)}{\Gamma(i\theta - (i-1)\alpha)}.$$

Corollary

Suppose that $R(x) = e^{\theta x}$, $\theta > 0$ *and* $\psi'(0+) \geq 0$. *Then the sequence* $W_n(x)$, $x > b$ *is given by* $W_0(x) = 1$ *and*

$$W_n(x) = \frac{1}{e^{n\theta x} \prod_{j=1}^{n} \psi(j\theta)}.$$

More convergence results

Using the Laplace transforms, we can show that

- if $R(x) = x^\theta$, $\psi(\lambda) = \lambda^\alpha$ with $1 < \alpha \le 2$ and $\theta > \alpha$, then the distribution of $T_b^-/\mathbb{E}_\infty T_b^-$ under $\mathbb{P}_\infty$ does not depend on b.

$$\mathbb{E}_\infty e^{-\frac{sT_b^-}{\mathbb{E}_\infty T_b^-}} = \left[\sum_{n=0}^{\infty} \left(\frac{s\Gamma(\theta)}{\Gamma(\theta-\alpha)}\right)^n \prod_{i=1}^{n} \frac{\Gamma(i\theta - i\alpha)}{\Gamma(i\theta-(i-1)\alpha)}\right]^{-1}.$$

- if there exists $1 < \alpha \le 2$ such that $\psi(\lambda) \sim \lambda^\alpha$ as $\lambda \to 0+$, and $R(x) = x^\theta$ for $\theta > \alpha$ and all x large enough, then $T_b^-/\mathbb{E}_\infty T_b^-$ converges in distribution under $\mathbb{P}_\infty$.
- if $R(x) = e^{\theta x}$ for $\theta > 0$ and $\psi'(0+) \ge 0$, then the distribution of $T_b^-/\mathbb{E}_\infty T_b^-$ under $\mathbb{P}_\infty$ does not depend on b.

$$\mathbb{E}_\infty e^{-\frac{sT_b^-}{\mathbb{E}_\infty T_b^-}} = \left[\sum_{n=0}^{\infty} \frac{(\psi(\theta)s)^n}{\prod_{j=1}^n \psi(j\theta)}\right]^{-1}.$$

Small time behavior of $\underline{X}_t$ in the slow regime

Recall that $g(x) := \mathbb{E}_\infty T_x^-$, $x > 0$ and g^{-1} is the inverse function of g.

Lemma

Suppose that process X comes down from infinity and $T_x^-/g(x) \to 1$ in probability as $x \to \infty$ under $\mathbb{P}_\infty$, and for any $h > 1$

$$\liminf_{x\to\infty} \frac{g(x)}{g(hx)} = c_h > 1.$$

Then we have

$$\frac{\underline{X}_t}{g^{-1}(t)} \to 1 \;\; \text{in } \mathbb{P}_\infty\text{-probability as } \; t \to 0+ \, .$$

Small time behavior of $\underline{X}_t$ in the fast regime

Lemma

Suppose that process X comes down from infinity and $T_x^-/g(x)$ converges in distribution under $\mathbb{P}_\infty$ and for any $h > 1$,

$$\liminf_{x\to\infty} \frac{g(x)}{g(hx)} = \infty.$$

Then

$$\frac{\underline{X}_t}{g^{-1}(t)} \to 1 \ \text{ in } \mathbb{P}_\infty\text{-probability as } \ t \to 0+ .$$

Two different ways of coming down

Proposition

Suppose that process X comes down from infinity and $\psi'(0+) = \frac{1}{W(\infty)} > 0$. Then $\mathbb{P}$-a.s., $\lim_{t \to 0+} \frac{\underline{X}_t}{X_t} = 1$.

Proposition

Suppose that process X comes down from infinity and for $\alpha' > 0$, $W(x) \sim x^{\alpha'}$ as $x \to \infty$, which by Tauberian theorem is equivalent to $\psi(\lambda) \sim \lambda^{-\alpha' - 1}$ as $\lambda \to 0+$. Then $\limsup_{t \to 0+} \frac{X_t}{\underline{X}_t} = \infty$.

In summary,

- if X is subcritical and comes down from infinity, then X_t comes down from infinity according to a deterministic function plus relatively small random fluctuations;
- if X comes down from infinity and $W(\infty) < \infty$, then X_t comes down from infinity with large random fluctuations

Given

$$\psi(\lambda) = b\lambda + \frac{\sigma^2\lambda^2}{2} + \lambda^\alpha,$$

- if $R(x) = x^\theta$, then
  - for $b > 0$ and $\theta > 1$,

$$\frac{X_t}{(b(\theta-1)t)^{1/(1-\theta)}} \to 1 \text{ in } \mathbb{P}_\infty\text{-probability as } t \to 0+.$$

  - for $b = 0 = \sigma^2$ and $\theta > \alpha$,

$$\frac{X_t}{(\frac{\Gamma(\theta)}{\Gamma(\theta-\alpha)}t)^{1/(\alpha-\theta)}} \to 1 \text{ in } \mathbb{P}_\infty\text{-probability as } t \to 0+.$$

- if $R(x) = e^{\theta x}$, then for $\theta > 0, b > 0$,

$$\frac{X_t}{\theta^{-1}\ln t^{-1}} \to 1 \text{ in } \mathbb{P}_\infty\text{-probability as } t \to 0+.$$

Thank you for your attention!